11 Math Cheat Sheet

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Contents

Part I

Sets and Functions

1 Sets

A well-defined collection of objects.

1.1 Representing sets

- In words
- Roster/tabulation method
- Rule method

1.2 Special types of sets

- Empty/null set $\emptyset$
- Singleton (has a single element)

1.3 Properties of sets

1.3.1 Cardinal number

Number of elements in a finite set. The number of elements in a set A is denoted by $n(A)$.

1.3.2 Equality

Exactly the same elements

1.3.3 Equivalence

Same number of elements

1.3.4 Overlapment

Having at least one element common

1.3.5 Disjoincy

Having no elements in common

1.4 Subsets

If A and B are sets such that every member of set A is a member of set **B,** A **is called a subset of** B **(**$A \subseteq B$**).**

1. B is a subset of B ($B \subseteq B$).
2. $\emptyset$ is a subset of all sets
3. For any two sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

If A and B are sets such that every member of set A is a member of set B, but set A is not equal to set B, A is called a proper subset of B ($A \subset B$). B is not a subset of B.

A set of sets is called a family of sets.

The set of all subsets of a set is called a power set. The power set of a set A is denoted as $P(A)$. A set having n elements has 2^n subsets.

1.5 Universal set

The set of all possible elements in a certain context is called the universal set and is denoted by ξ or U or E.

1.6 Operations on sets

1.6.1 Union of sets

$$A \cup B = \{x|(x \in A||x \in B)\}$$

1. $A \cup B = B \cup A$

2. $(A \cup B) \cup C = A \cup (B \cup C)$

3. $A \subseteq A \cup B$ and $B \subseteq A \cup B$

4. If $A \subseteq B$, $A \cup B = B$

5. $A \cup \emptyset = A$

1.6.2 Intersection of sets

$$A \cap B = \{x | (x \in A \&\& x \in B)\}$$

1. $A \cap B = B \cap A$

2. $(A \cap B) \cap C = A \cap (B \cap C)$

3. If $A \subseteq B$, $A \cap B = A$

4. $A \cap \emptyset = \emptyset$

1.6.3 Difference of sets

If A and B are sets, $A - B$ is the set of elements in A and not in B and $B - A$ is the set of elements in B and not in A.

1.6.4 Symmetric difference

$$A \Delta B = (A - B) \cup (B - A)$$

$$A \Delta B = (A \cup B) - (B \cap A)$$

$$A \Delta B = B \Delta A$$

$$A \Delta A = \emptyset$$

1.6.5 Complement of a set

The complement of a set A is the set of all elements which do not belong to A. It is denoted as A' or $\overline{A}$ or A^e.

1. $(A')' = A$
2. $\xi' = \emptyset$ and $\emptyset' = \xi$
3. $A \cup A' = \xi$ and $A \cap A' = \emptyset$
4. $(A \cup B)' = A' \cap B'$ or $(A \cap B)' = A' \cup B'$

1.7 Cardinal number of complex sets

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B \cup C) = n(a) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

2 Relations and Functions

Part II

Trigonometry

3 Angles and Arc Lengths

$$(2\pi)^r = 360^\circ$$

$$\theta = \frac{s}{r}$$

where θ is in radians, s is the length of the arc subtended by θ and r is the radius of the circle.

$$A = \frac{r^2\theta}{2} = \frac{rs}{2}$$

where A is the area of a sector with angle θ.

4 Trigonometrical Functions

The following table can be derived by visualizing the unit circle for each of these angles.

	$-\theta$	$90^\circ - \theta$	$90^\circ + \theta$	$180^\circ - \theta$	$180^\circ + \theta$	$360^\circ - \theta$	$360^\circ + \theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\tan\theta$	$\tan\theta$	$\tan\theta$	$\tan\theta$

5 Compound and Multiple Angles

5.1 Sum

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan A \tan C}$$

5.2 Difference

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot A - \cot B}$$

5.3 Product to Sum

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$2 \cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

5.4 Sum to Product

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

5.5 Double Angle

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$2\sin^2 A = 1 - \cos 2A$$

$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

$$2\cos^2 A = 1 + \cos 2A$$

5.6 Half Angle

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2} = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

5.7 Triple Angle

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

5.8 One-third Angle

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$$

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$$

$$\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}}$$

5.9 Miscellaneous

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$$

$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$$

6 Trigonometric Equations

Equation	General solution
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = (2n+1)\pi$
$\tan\theta = 0$	$\theta = n\pi$
$\cot\theta = 0$	$\theta = (2n+1)\pi$
$\sec\theta = 0$	—
$\csc\theta = 0$	—
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\theta = n\pi \pm \alpha$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$

7 Properties of Triangle

7.1 Area of Triangle

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$$

7.2 Sine Rule

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin B}{B}$$

7.3 Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = c^2 + a^2 - 2ca\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

8 Mathematical Induction

A statement $T(n)$ is true for all $n \in \mathbb{N}$, provided $T(1)$ is true and $T(k) \Rightarrow T(k+1)$.

Steps to be followed:

1. Basic step: Show $T(1)$ is true.
2. Assume that $T(k)$ is true.
3. Induction step: Prove that that implies that $T(k+1)$ is true.

Part III

Algebra

9 Complex Numbers

The basis of imaginary numbers is the imaginary unit, i, or iota such that

$$i^2 = -1$$

$$\Rightarrow i = \sqrt{-1}$$

Complex numbers are numbers of the form $a+bi$ where a and b are real numbers. a is the real part of the complex number and is denoted as $Re(a+bi)$. bi is the imaginary part of the complex number and is denoted as $Im(a+bi)$.

9.1 Powers of i

The value of whole number powers of i can be found easily as there is a simple pattern to them.

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = 1$$

Subsequent powers just cycle over these same values. In an attempt to generalize i^n, where n is a whole number, we get $i^n = (i^4)^k \cdot i^{n-4k} = i^{n-4k}$. What we want, is the remainder when n is divided by 4, that is, $n \mod 4$. Therefore,

$$i^n = i^{n \mod 4}$$

9.2 Graphing complex numbers

Complex numbers are graphed in the complex plane or the argand plane. The plane has two axes, a horizontal axis, the real axis, and a vertical axis, the imaginary axis.

9.3 Operations on complex numbers

For two complex numbers $z_1 = a + bi$ and $z_2 = c + di$,

$$z_1 + z_2 = (a + c) + (b + d)i$$

$$-z_1 = -a + (-b)i$$

$$z_1 - z_2 = (a - c) + (b - d)i$$

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

9.4 Conjugate

The conjugate of an complex number $z = a + bi$ is another complex number $\overline{z} = a - bi$. If z is graphed in the complex plane, $\overline{z}$ is the reflection of z in the x-axis.

9.5 Modulus

If a complex number z is denoted in the complex plane, its modulus, $|z|$ is the distance of z from the origin of the complex plane. If $z = a + bi$,

$$|z| = \sqrt{a^2 + b^2}$$

9.5.1 Properties

$$z = 0 \iff |z| = 0$$

$$z\overline{z} = |z|^2$$

$$|z| = |\overline{z}|$$

$$-|z| \leq \mathrm{Re}(z) \leq |z|$$

$$-|z| \leq \mathrm{Im}(z) \leq |z|$$

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

$$|z_1 z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

9.6 Polar form

For a complex number $z = a + bi$,

$$a = r\cos\theta$$

$$b = r\sin\theta$$

Thus, $z = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta) = r\mathrm{cis}\theta$. This is the polar form of z.

$$\tan\theta = \frac{b}{a}$$

$$\theta = \tan^{-1}\frac{b}{a}$$

$$r = \sqrt{a^2 + b^2} = |z|$$

The angle θ is also called the argument or amplitude of $z = a + bi$, denoted as $\theta = \mathrm{arg}z = \mathrm{am}z$.

$$\mathrm{am}(z_1 z_2) = \mathrm{am}z_1 + \mathrm{am}z_2$$

$$\mathrm{am}(\frac{z_1}{z_2}) = \mathrm{am}z_1 - \mathrm{am}z_2$$

9.7 Cube roots of unity

$$\sqrt[3]{1} = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

If one of the complex cube roots is denoted as ω, the other is ω^2.

$$\omega = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\sqrt[3]{1} = 1, \omega, \omega^2$$

$$1 + \omega + \omega^2 = 0$$

$$\omega^3 = 1$$

10 Quadratic Equations

A quadratic equation is a polynomial with degree 2. In the cartesian plane, the graph of a quadratic equation is a parabola. Forms of a quadratic equation:

- Standard form
- Intercept form
- Vertex form

If $f(x)$ is a quadratic function, the roots of the equation are values k such that $f(k) = 0$. Quadratic equations have zero, one, or two real roots. The following terms are synonyms of roots:

- solutions
- zeroes(because $f(root) = 0$)
- x-intercepts(because the roots are the x-intercepts of the graph)

The standard form is $ax^2 + bx + c = 0$ where a, b and c are constants. The standard form is the least useful of them all. When a is positive, the graph of the equation is upward opening. When it is negative, the graph is downward opening. The axis of symmetry of the parabola is the line $x = \frac{-b}{2a}$.

The intercept form is $a(x-\alpha)(x-\beta)=0$ where a is a constant and α and β are the roots of the equation. This form clearly shows the roots of the equation.

The vertex form is $a(x-h)^2+k=0$ where (h,k) is the vertex of the graph of the equation. This form is useful for determining maximum/minimum value of the function. The direction of the parabola can be determined from the sign of a like it is determined for the standard form. The axis of symmetry of the parabola is the line $x=h$.

10.1 Determining nature of roots

The nature of the roots of a quadratic equation can be determined without solving for the roots. For a quadratic equation in the standard form, the discriminant, Δ, is given as $\Delta = b^2-4ac$ where a, b and c have there usual meanings. The nature of the roots $x_1$and x_2 is given as

$$x_1, x_2 \begin{cases} x_1, x_2 \in \mathbb{Q} & \Delta > 0, \sqrt{\Delta} \in \mathbb{Z} \\ x_1, x_2 \in \mathbb{R} & \Delta > 0, \sqrt{\Delta} \notin \mathbb{Z} \\ x_1 = x_2 & \Delta = 0 \\ x_1, x_2 \in \mathbb{C}, x_1, x_2 \notin \mathbb{R} & \Delta < 0 \end{cases}$$

10.2 Quadratic formula

The roots of a quadratic equation in the standard form can be determined using the formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Where a, b and c have there usual meanings.

10.3 Determining the range of roots

10.4 Equations reducible to quadratic equations

10.4.1 $ax^{2n} + bx^n + c = 0$

Let $y = x^{2n}$ and solve as a quadratic.

10.4.2 $az + \frac{b}{z} + c = 0$

Multiply the whole equation by z and solve as a quadratic.

10.4.3 $(x + a)(x + b)(x + c)(x + d) + k = 0$

When the sum of two of a, b, c, d is equal to the sum of the other two, the equation can be reduced to a quadratic.

Say $a + d = b + c = m$.

$[(x + a)(x + d)]\,[(x + b)(x + c)] + k = 0$

$\left[x^2 + mx + ad\right]\left[x^2 + mx + bc\right] + k = 0.$

Let $x^2 + mx$ be y.

Now solve $(y + ad)(y + bc) + k = 0$ as a quadratic.

10.5 Sums and products of roots

For a given quadratic equation $ax^2 + bx + c = 0$ with roots α and β,

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

11 Inequalities

12 Permutations and Combinations

The number of ways to choose and arrange r out of n items is given as:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Number of ways to arrange n items where p_0 are of one kind, p_1 are of another kind etc is given as:

$$\frac{n!}{\Pi p_i}$$

Number of circular arrangements of n different things:

$$(n-1)!$$

Number of circular arrangements of n different things where clockwise and counter-clockwise count as the same:

$$\frac{1}{2}(n-1)!$$

Number of circular arrangements of n different things taken r at a time:

$$\frac{P(n, r)}{r}$$

Number of circular arrangements of n different things taken r at a time where clockwise and counter-clockwise count as the same:

$$\frac{1}{2} \times \frac{P(n, r)}{r}$$

Number of ways to choose r out of n items:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Number of diagonals in a polygon:

$$C(n,2) - n$$

$$C(n,r) + C(n,r-1) = C(n+1,r)$$

where $1 \leq r \leq n$.

13 Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Similarly,

$$(a-b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-b)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k} b^k$$

The kth term term in the expansion of $(a+b)^n$, T_k is given as

$$T_k = \binom{n}{k-1} a^{n-k+1} b^{r-1}$$

13.1 Properties of Binomial Expansions

For the binomial expansion of $(a+b)^n$,

- The number of terms is $n+1$.
- The sum of the indices of a and b in each term is n.
- The sum of binomial coefficients is 2^n.

14 Sequence and Series

14.1 Arithmetic Progressions

Arithmetic progressions(APs) are progressions where each term differs from the previous one by a constant value. This constant value is called the common difference.

The nth term of an AP is given as

$$T_n = a + d(n-1)$$

where a is the first term of the sequence and d is the common difference.

The sum of the first n terms of an AP is given as

$$S_n = \frac{n}{2}(a+l) = \frac{n}{2}(2a + d(n-1))$$

where l is the last term of the sequence and a and d have their usual meanings.

14.2 Geometric Progressions

Geometric progressions(GPs) are sequences where each term can be found by multiplying the preceding term by a constant value which is neither zero, nor one. The constant value is called the common ratio.

The nth term of a GP is given as

$$T_n = ar^{n-1}$$

where a is the first term of the GP and r is the common ratio.

The sum of the first n terms of a GP is given as

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a-lr}{1-r}$$

where all symbols hold their usual meanings. Some publishers give the sum as

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r} = \frac{a-lr}{1-r} & r < 1 \\ \frac{a(r^n-1)}{r-1} = \frac{lr-a}{r-1} & r > 1 \end{cases}$$

However, all the above expressions are equivalent.

The sum of the terms of an infinite GP exists only when $|r| < 1$. It is given as

$$S_\infty = \frac{a}{1-r}$$

14.3 Arithmetico-geometric Series

A series each term of which is found by multiplying the corresponding terms of an AP and a GP is called an arithmetico-geometric(AG) series. S_n is the sum of the first n terms of an AG series while S_∞ is the sum of the AG series to infinity.

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+d(n-1))r^n}{1-r}$$

where a is the first term of the AP, d is the common difference of the AP and r is the common ratio of the GP. The above result is valid only when the GP is of the form $1, r, r^2 \ldots$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

where a, d and r hold the same meanings as above.

14.4 Series involving natural numbers

Expressions for the sum of powers of the first n natural numbers can be derived.

Sum of first n natural numbers is given as

$$\sum n = \frac{n(n+1)}{2}$$

Sum of the squares of the first n natural numbers is given as

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of the cubes of the first n natural numbers is given as

$$\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum n\right)^2$$

Part IV

Coordinate Geometry

15 Basic Concepts of Points and their Coordinates

The distance d between two points (x_1, y_1) and (x_2, y_2) is given as:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The coordinates of the point that divides the distance between (x_1, y_1) and (x_2, y_2) *internally* in the ratio $m : n$ are:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

The coordinates of the point that divides the distance between (x_1, y_1) and (x_2, y_2) *externally* in the ratio $m : n$ are:

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

The coordinates of the centroid of a triangle with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) is given as:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The area Δ of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is given as:

$$\Delta = \pm\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

16 The Straight Line

16.1 Standard Form

$$ax + by + c = 0$$

16.2 Slope-Intercept Form

$$y = mx + c$$

where m is the slope of the line and c is the y-intercept of the line.

16.3 Angle Between Two lines

$$\tan\theta = \left|\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}\right| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where θ_1 and θ_2 are the angles the lines make with the positive end of the x-axis and m_1 and m_2 are the slopes.

16.4 Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a is the x-intercept and b is the y-intercept.

16.5 Normal Form

$$x\cos\alpha + y\sin\alpha = p$$

where α is the angle the line makes with the positive end of the x-axis and p is the perpendicular distance of the line from the origin.

16.6 Perpendicular distance between a line and a point

The perpendicular distance between $ax + by + c = 0$ and (x_1, y_1) is given as

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

16.7 Angular Bisector

The equation of the angle bisectors of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$\frac{a_1x + b_1y + c}{\sqrt{a_1^2 + b_!^2}} = \pm\frac{a_2x + b_2y + c}{\sqrt{a_2^2 + b_2^2}}$$

16.8 Lines parallel to a line

The equation of the lines parallel to $ax + by + c = 0$ is given as

$$ax + by + k = 0$$

16.9 Lines perpendicular to a line

The equation of the lines perpendicular to $ax + by + c = 0$ is given as

$$-bx + ay + k = 0$$

17 Circle

$$(x - h)^2 + (y - k)^2 = r^2$$

Centre: (h, k), radius: r

$$(x - h)^2 + (y - k)^2 = r^2 \Rightarrow x^2 + y^2 - 2hx - 2kx + h^2 + k^2 - r^2 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where $g = -k$, $f = -k$ and $c = h^2 + k^2 - r^2$. Centre: $(-g, -f)$, radius: $\sqrt{g^2 + f^2 - c}$

If the ends of the diameter of a circle are (x_1, y_1) and (x_2, y_2), the equation of the circle is given as

$$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

17.1 Intercept on the axes

The x and y intercepts can be found by plugging in $y = 0$ and $x = 0$ respectively and solving the resulting quadratics.

17.2 Intersection of a line and a circle

1. Plug in the value of x or y from the equation of the line into the equation of the circle
2. Solve the resulting quadratic for the two values of the variable
3. Plug in the resulting values into the *equation of the line* and solve to get the values of the other variable

The pairs of x and y co-ordinates are the points of intersection of the line and the circle.

17.3 Equation of a tangent

17.3.1 At a point on the circle

1. Find the slope of the radius through the point of contact
2. Slope of the tangent is its negative reciprocal
3. Use $y - y_1 = m(x - x_1)$

17.3.2 With a given slope m

1. Slope of the radius perpendicular to the tangent is the negative reciprocal of m
2. Let the equation of the tangent be $y = mx + b$
3. Find the perpendicular distance between the equation of the tangent and the centre, set it equal to the circle's radius and solve

17.3.3 Through an external point (x_1, y_1)

1. Let the equation of the tangent be $y - y_1 = m(x - x_1)$
2. Find the perpendicular distance between the equation of the tangent and the centre, set it equal to the circle's radius and solve

17.3.4 Of $x^2 + y^2 = a^2$ in terms of the tangent's slope

$$y = mx \pm 4\sqrt{1 + m^2}$$

where m is the slope of the tangent

Part V

Calculus

18 Limits

Given a number $\delta > 0$ howsoever small, if x takes up values such that $0 < |x - a| < \delta$ then x is said to tend to a and is symbolically written $x \to a$.

A function $f(x)$ is said to tend to a limit l when x tends to a if the difference between $f(x)$ and l and can be made as small as we please by making x sufficiently close to a.

$$\lim_{x \to a} f(x) = l$$

If for every positive number ϵ, however small it may be, there exists a positive number δ such that whenever $0 < |x - a| < \delta$ we have $|f(x) - 1| < \epsilon$ then we say $f(x)$ tends to the limit l as 'x tends to a'.

A function will have a limit iff the limit as x approaches a from the right side is equal to the limit as x approaches a from the left side, that is, iff the right limit is equal to the left limit.

Theorems

$$\lim_{x \to a} (f + g)x = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (f - g)x = \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (fg)x = \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (f \div g)x = \lim_{x \to a} [f(x) \div g(x)] = \lim_{x \to a} f(x) \div \lim_{x \to a} g(x)$$

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to 0}]^n$$

where n is a positive integer

If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, $\lim_{x\to a} \frac{1}{f(x)} = 0$

$$\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right)$$

$$\lim_{x\to a} f \circ g(x) = f\left(\lim_{x\to a} g(x)\right)$$

18.1 Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \times \infty$
- $\infty - \infty$
- 1^∞
- ∞^0
- ...

18.2 Algebraic limits

Often declaring $x = a + h$ is useful because then when $x \to a$, $h \to 0$.

18.2.1 Direct substitution

If on directly substituting a value in the given expression the result is finite, that finite number is the limit of the given expression.

18.2.2 Factorization

18.2.3 Rationalization

In functions that involve roots, rationalization of the numerator or denominator can help

18.2.4 Using expansions

- Binomial expansion
    - Stuff cancels out
        - * \o/

18.2.5 $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}$ where $a > 0$

18.2.6 $\lim_{x\to a}\left(\frac{x^m-a^m}{x^n-a^n}\right) = \frac{m}{n}a^{m-n}$ where $a > 0$

18.3 Infinite limits

18.3.1 Divide fraction by highest power of x

$\frac{1}{x}$, $\frac{1}{x^2}$, ... $\to 0$ as $x \to \infty$

18.3.2 Set $x = \frac{1}{y}$. As $x \to \infty$, $y \to 0$

18.4 Trigonometric limits

1. $\lim_{x\to 0} \sin x = 0$
2. $\lim_{x\to 0} \cos x = 1$
3. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
4. $\lim_{x\to 0} \frac{\tan x}{x} = 1$
5. $\lim_{x\to \infty} \frac{\sin x}{x} = 0$
6. $\lim_{x\to \infty} \frac{\cos x}{x} = 0$

18.4.1 When $x \to a$ $(a \neq 0)$

If $x = a + h$, as $x \to a$, $h \to 0$.

18.4.2 Factorization

Use them trig formulae

18.5 Exponential and logarithmic limits

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e^x$$

$$\lim_{x\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 1}\left(\frac{\log x}{x-1}\right) = 1$$

18.6 L'Hôpital's rule

If $\frac{f(a)}{g(a)}$ is of indeterminate form, $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

19 Differentiation

The derivative of $f(x)$ is denoted by the following (equivalent) notations:

- $\frac{dy}{dx}$
- $\frac{d}{dx}y$
- $\frac{d}{dx}f(x)$
- Dy
- y'
- f' or $f'(x)$

It is defined as:

$$\frac{d}{dx}f(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

Solving a derivative using the above equation is called "solving using first principle".

19.1 Rules

- Power rule
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
- Chain rule
$$\frac{d}{dx}(f \circ g) = \frac{d}{dx}(\frac{d}{dg}(f)) = f' \circ g \cdot g'$$
- Product rule
$$\frac{d}{dx}(f \cdot g) = f'g + fg'$$
- Quotient rule
$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

19.2 Some Important Derivatives

$$\frac{d}{dx}(k) = 0$$

$$\frac{d}{dx}(a^x) = \frac{a^x}{\ln a}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Part VI

Measures of Dispersion

20 Measures of Central Tendency

20.1 Basic

$$\overline{x} = \frac{\sum x_i}{n}$$

where n is the number of elements.

20.2 Direct method

$$\overline{x} = \frac{\sum(fx)}{\sum f}$$

where f is the frequency of the value x. This method is meant to be used for grouped data.

20.3 Short-cut method

$$\overline{x} = A + \frac{\sum(fd)}{\sum f}$$

where $d = x - A$

20.4 Step deviation method

$$\overline{x} = A + \frac{\sum(fu)}{\sum f} \times i$$

where $u = \frac{x-A}{i}$.

21 Measures of Dispersion

Mean deviation d is:

$$d = \frac{\sum |x - \overline{x}|}{n}$$

The standard deviation is:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

The variance is:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2$$

22 Probability

$$\text{Odds in favour of A } = \frac{n(A)}{1 - n(A)} = \frac{P(A)}{1 - P(A)}$$

$$\text{Odds against A } = \frac{1 - n(A)}{n(A)} = \frac{1 - P(A)}{P(A)}$$

Part VII

Conic Section

23 Parabola

24 Ellipse

25 Hyperbola

Part VIII

Introduction to 3-Dimensional Geometry

26 Points and their Coordinates in 3-Dimensions

27 Mathematical Reasoning

Part IX

Statistics

28 Statistics(Continued)

Combined mean is given as

$$\overline{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_1 + n_2}$$

Combined standard deviation is given as

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where n_1 and n_2 are the number of observations in either distribution, σ_1 and σ_2 are the standard deviations, and d_1 and d_2 are the deviation of either mean from the respective combined mean ($d_n = |\overline{x} - \overline{x_n}|$).

$$M_d = Q_2 = l + \frac{i}{f}\left(\frac{n}{2} - c\right)$$

where l is the lower limit of the median class, i is the class size, f is the frequency of the median class, n is the number of observations in the distribution, and c is the cumulative frequency of the class preceding the median class.

Similarly,

$$Q_1 = l + \frac{i}{f}\left(\frac{n}{4} - c\right)$$

$$Q_3 = l + \frac{i}{f}\left(\frac{3n}{4} - c\right)$$

$$D_k = l + \frac{i}{f}\left(\frac{kn}{10} - c\right)$$

$$P_k = l + \frac{i}{f}\left(\frac{kn}{100} - c\right)$$

The mode of a grouped distribution is given as

$$l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}}$$

where l is the lower limit of the modal class, f_m is the frequency of the modal class, f_{m-1} is the frequency of the class preceding the modal class, and f_{m+1} is the frequency of the class following the modal class.

29 Correlation Analysis

29.1 Covariance

$$cov(x, y) = \frac{(x_1 - \overline{x})(y_1 - \overline{y}) + \ldots + (x_n - \overline{x})(y_n - \overline{y})}{n} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{n} = \frac{\sum_{i=1}^{n} d_x d_y}{n}$$

29.2 Karl Pearson's Coefficient of Correlation

$$r = \frac{\sum d_x d_y}{\sqrt{\sum d_x^2 \sum d_y^2}}$$

where r is Karl Pearson's coefficient of correlation, $d_x = x_i - \overline{x}$, and $d_y = y_i - \overline{y}$.

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{(\sum x)^2}{n}} \times \sqrt{\sum y^2 - \frac{(\sum y)^2}{n}}}$$

The above formula is used only when values of x and y are small.

$$\frac{\sum uv - \frac{\sum u \sum v}{n}}{\sqrt{\sum u^2 - \frac{(\sum u)^2}{n}} \times \sqrt{\sum v^2 - \frac{(\sum v)^2}{n}}}$$

where $u = x - A_1$ (deviation taken from assumed mean A_1 of x series), $v = y - A_2$ (deviation taken from assumed mean A_2 of y series). The above formula is used when $\overline{x}$ and/or $\overline{y}$ have decimal parts.

29.3 Spearman's Rank Correlation Coefficient

$$R = 1 - \frac{6\sum D^2}{n(n^2-1)}$$

where D is the difference between the corresponding ranks of corresponding items of the two series and n is the number of individuals in each series.

When there are ties in a set of data, we use a correction factor.

$$R = 1 - \frac{6\left[\sum D^2 + \frac{1}{12}\sum(m_i^3 - m)\right]}{n(n^2-1)}$$

where m_i is the number of times a repeated value is repeated.

30 Index Numbers

30.1 Unweighted

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100$$

$$P_{01} = \frac{\sum x}{n}$$

where $x = \frac{\sum P_1}{\sum P_0} \times 100$.

30.2 Weighted

$$P_{01} = \frac{\sum WP_1}{\sum WP_0} \times 100$$

$$P_{01} = \frac{\sum(Wx)}{\sum W}$$

where $x = \frac{P_1}{P_0} \times 100$

31 Moving Average

Part X

Non-ISC chapters

32 Logarithms

If $a^b = c$, $\log_a c = b$. $\log_a c$ is undefined when $a < 0$, $a = 1$ or $c < 0$.

32.1 Laws

1. $\log_x x = 1$
2. $\log_x 1 = 0$
3. $\log_a b = \frac{1}{\log_b a}$
4. $\log_x a + \log_x b = \log_x ab$
5. $\log_x a - \log_x b = \log_x \frac{a}{b}$
6. $-\log_x a = \log_x \frac{1}{a}$
7. $\log_x a^b = b \log_x a$
8. $\log_{x^y} a = \frac{1}{y} \log_x a$
9. $\log_{x^y} a^b = \frac{b}{y} \log_x a$
10. $\log_b a = \frac{\log_x a}{\log_x b}$ where x is any valid base
11. $a^{\log_a b} = b$
12. $a^{\log_x b} = b^{\log_x a}$