# 12 Math Cheat Sheet

Shreyas Minocha

# Contents


**I Relations and Functions** **8**

**1 Relations** **8**

1.1 Reflexive . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

1.2 Symmetric . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

1.3 Transitive . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

1.4 Equivalence . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

**2 Functions** **9**

2.1 One-one (injection) . . . . . . . . . . . . . . . . . . . . . . . . 9

2.2 Onto (surjection) . . . . . . . . . . . . . . . . . . . . . . . . . 9

2.3 Bijection . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

2.4 Composition . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

2.5 Inverse . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

**3 Binary Operations** **10**

3.1 Properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

3.1.1 Commutative . . . . . . . . . . . . . . . . . . . . . . . . 10

3.1.2 Associative . . . . . . . . . . . . . . . . . . . . . . . . . 10

3.1.3 Distributive Law . . . . . . . . . . . . . . . . . . . . . . 10

3.2 Identity Element . . . . . . . . . . . . . . . . . . . . . . . . . . 10

**4 Inverse Trigonometric Functions** **11**

**II Algebra** **14**

**5 Determinants** **14**

**6 Matrices** **16**

6.1 Adjoint . . . . . 16

6.2 Inverse . . . . . 16

6.3 Inverse by elementary operations . . . . . 16

6.4 Properties . . . . . 16

6.5 Area of a triangle . . . . . 17

6.6 Symmetric and skew-symmetric matrices . . . . . 17

6.7 Solving equations . . . . . 17

6.7.1 Conditions . . . . . 17

6.7.2 2 variables, 2 equations . . . . . 17

6.7.3 3 variables, 3 equations . . . . . 18

**III Calculus** **19**

**7 Continuity and Differentiability of Functions** **19**

7.1 Conditions For Continuity . . . . . 19

7.2 Conditions For Differentiability . . . . . 19

**8 Differentiation** **20**

**9 Indeterminate Forms of Limits** **23**

**10 Mean Value Theorems** **24**

10.1 Rolle's Theorem . . . . . 24

10.2 Langrange's Mean Value Theorem . . . . . 24

**11 Applications of Derivatives** **25**

11.1 Tangents and Normals . . . . . 25

11.2 Approximations . . . . . 25

11.3 Rate Measuring . . . . . 25

11.4 Monotonocity . . . . . 25

**12 Maxima and Minima** **26**

**13 Indefinite Integrals — Standard Forms** **27**

**14 Indefinite Integrals — Methods of Integration** **28**

**15 Indefinite Integrals — Special Integrals** **29**

15.1 $\int \sqrt{ax^2+bx+c}\mathrm{d}x$, $\int (px+q)\sqrt{ax^2+bx+c}\mathrm{d}x$ . . . . . 29

15.2 $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\mathrm{d}x$ . . . . . 29

15.3 $\int \frac{x^2\pm 1}{x^4+kx^2+1}\mathrm{d}x$ ($k$ may be 0) . . . . . 29

15.4 $\int \frac{\mathrm{d}x}{a+b\cos x}$, $\int \frac{\mathrm{d}x}{a+b\sin x}$, $\int \frac{\mathrm{d}x}{a\cos x+b\sin x}$, $\int \frac{\mathrm{d}x}{a\cos x+b\sin x+c}$ . . . . . 30

15.5 $\int \frac{\mathrm{d}x}{a\cos^2 x+b\sin^2 x+c}$ . . . . . 30

15.6 $\int \frac{\mathrm{d}x}{1+x^4}$ . . . . . 30

15.7 $\int \sqrt{\tan x}\mathrm{d}x$, $\int \sqrt{\cot x}\mathrm{d}x$ . . . . . 30

**16 Definite Integrals** **31**

**17 Definite Integral as a limit of a sum** **31**

**18 Differential Equations** **33**

18.1 Variable Separable . . . . . 33

18.1.1 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ . . . . . 33

18.1.2 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$ . . . . . 33

18.1.3 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \cdot \phi(y)$ . . . . . 33

18.1.4 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c)$ . . . . . 33

18.2 Reducible to Variable Separable . . . . . 33

18.3 Homogeneous equations of first order . . . . . 33

18.3.1 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x,y)}{g(x,y)} = m\left(\frac{y}{x}\right)$ . . . . . 33

18.4 Linear equations . . . . . 33

18.4.1 $\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$ . . . . . 33

**IV Probability** **34**

**19 Probability** **34**

19.1 $A$ and $B$ are mutually exclusive . . . . . 34

19.2 $A$ and $B$ are independent events . . . . . 34

19.3 Geometric Progression . . . . . 35

**20 Bayes' Theorem** **36**

**21 Theoretical Probability Distribution** **37**

21.1 Binomial Distribution . . . . . 37

**V Vectors** **38**

**22 Vectors** **38**

**23 Vectors (Continued)** **38**

**VI Three-Dimensional Geometry** **38**

**24 Three-Dimensional Geometry** **38**

**25 The Plane** **38**

**VII Application of Integrals** **38**

**26 Areas of a Curve** **38**

**VIII Application of Calculus** **39**

**27 Application of Calculus in Commerce and Economics** **39**

**IX Linear Regression** **41**

**28 Linear Regression** **41**

**X Linear Programming** **42**

**29 Linear Programming** **42**

**XI Appendices** **43**

**A Useful Trigonometric Formulae** **43**

**B Other Useful Formulae** **43**

## Part I

# Relations and Functions

## 1 Relations

### 1.1 Reflexive

$$aRa$$

### 1.2 Symmetric

$$aRb \implies bRa$$

### 1.3 Transitive

$$aRb \wedge bRc \implies aRc$$

### 1.4 Equivalence

If a relation is *reflexive*, *symmetric* and *transitive*, it is an equivalence relation.

# 2 Functions

## 2.1 One-one (injection)

If $f(a) = f(b) \implies a = b$, $f(x)$ is a one-one function.

## 2.2 Onto (surjection)

A function $f : A \to B$ is onto if every element of $B$ has a pre-image in A.

## 2.3 Bijection

A function $f(x)$ is a bijection if it is both one-one and onto.

## 2.4 Composition

$$(f \circ g)(x) = f(g(x))$$

$$(g \circ f)(x) = g(f(x))$$

## 2.5 Inverse

- Function must be one-one.
- Horizontal line test.
- Reflection in $y = x$.
- If $f : A \to B$, $f^{-1} : B \to A$.

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

# 3 Binary Operations

$$\text{Number of binary operations} = n^{n^2}$$

## 3.1 Properties

### 3.1.1 Commutative

$$a * b = b * a$$

### 3.1.2 Associative

$$a * (b * c) = (a * b) * c$$

### 3.1.3 Distributive Law

$$a * (b \odot c) = (a * b) \odot (a * c)$$

## 3.2 Identity Element

$e \in S$ is the identity element wrt $*$ on $S$ if the following holds for every $x \in S$.

$$e * x = x * e = x$$

## 4 Inverse Trigonometric Functions

$$\csc^{-1} x = \sin^{-1} \frac{1}{x}$$

$$\sec^{-1} x = \cos^{-1} \frac{1}{x}$$

$$\cot^{-1} x = \begin{cases} \tan^{-1} \frac{1}{x} & x > 0 \\ \pi + \tan^{-1} \frac{1}{x} & x < 0 \end{cases}$$

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$\csc^{-1}(-x) = -\csc^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\cos^{-1}(-x) = \pi - \cos x$$

$$\sec^{-1}(-x) = \pi - \sec x$$

$$\cot^{-1}(-x) = \pi - \cot x$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

If $xy < 1$,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

If $xy > -1$,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$$

If $x^2 < 1$,

$$2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2}$$

$$2\sin^{-1} x = \sin^{-1} \left(2x\sqrt{1-x^2}\right)$$

$$2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

$$3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$$

$$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

$$3\tan^{-1} x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$$

If <conditions>,

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1} \left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

If <conditions>,

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1} \left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2} \cdot \sqrt{1-y^2}\right)$$

# Part II

# Algebra

## 5 Determinants

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \end{vmatrix} = 0$$

$$\begin{vmatrix} \longrightarrow & \longrightarrow & \longrightarrow \\ \longrightarrow & \longrightarrow & \longrightarrow \\ \longrightarrow & \longrightarrow & \longrightarrow \end{vmatrix} = \begin{vmatrix} \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \end{vmatrix}$$

$$\begin{vmatrix} a & d & \cdot \\ b & e & \cdot \\ c & f & \cdot \end{vmatrix} = - \begin{vmatrix} d & a & \cdot \\ e & b & \cdot \\ f & c & \cdot \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ a & b & c \\ \cdot & \cdot & \cdot \end{vmatrix} = 0$$

$$\begin{vmatrix} ka & kb & kc \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = k \begin{vmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$\begin{vmatrix} a+x & \cdot & \cdot \\ b+y & \cdot & \cdot \\ c+z & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a & \cdot & \cdot \\ b & \cdot & \cdot \\ c & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} x & \cdot & \cdot \\ y & \cdot & \cdot \\ z & \cdot & \cdot \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ g+ka & h+kb & i+kc \end{vmatrix} = \begin{vmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a+ld+mg & d & g \\ b+le+mh & e & h \\ c+lf+mi & f & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

# 6 Matrices

## 6.1 Adjoint

$$\operatorname{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\operatorname{adj}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}'$$

## 6.2 Inverse

$A$ has an inverse only if $A$ is non-singular ($|A| \neq 0$).

$$A^{-1} = \frac{\operatorname{adj}A}{|A|}$$

## 6.3 Inverse by elementary operations

$$A = IA$$

| | Row-wise | Column-wise |
|---|---|---|
| 1 | $R_i \longleftrightarrow R_j$ | $C_i \longleftrightarrow C_j$ |
| 2 | $R_i \longrightarrow kR_i$ | $C_i \longrightarrow kC_i$ |
| 3 | $R_i \longrightarrow R_i + kR_j$ | $C_i \longrightarrow C_i + kC_j$ |

## 6.4 Properties

$$|\operatorname{adj}A| = |A|^2$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A')^{-1} = (A^{-1})'$$

## 6.5 Area of a triangle

Vertices: $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

## 6.6 Symmetric and skew-symmetric matrices

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

## 6.7 Solving equations

### 6.7.1 Conditions

| $\lvert A\rvert$ | $(\text{adj}A)B$ | Singularity | Number of solutions | Consistency |
|---|---|---|---|---|
| $\lvert A\rvert \neq 0$ | N.A. | Non-singular | Unique solution | Consistent |
| $\lvert A\rvert = 0$ | $(\text{adj}A)B \neq 0$ | Singular | No solution | Inconsistent |
| | $(\text{adj}A)B = 0$ | | Infinitely many solutions | Consistent |

### 6.7.2 2 variables, 2 equations

$$a_1x + b_1y = p$$
$$a_2x + b_2y = q$$

$$\begin{bmatrix} a_1 & b_! \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

aka $AX = B$

$$X = A^{-1} \times B$$

#### 6.7.3 3 variables, 3 equations

$$a_1x + b_1y + c_1z = p$$
$$a_2x + b_2y + c_2z = q$$
$$a_3x + b_3y + c_3z = r$$

$$\begin{bmatrix} a_1 & b_! & c_! \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

aka $AX = B$

$$X = A^{-1} \times B$$

# Part III

# Calculus

## 7 Continuity and Differentiability of Functions

### 7.1 Conditions For Continuity

- $f(a)$ is defined
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$

### 7.2 Conditions For Differentiability

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$Lf'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

$f(x)$ is differentiable at $x = a$ if $Rf'(a) = Lf'(a)$

## 8 Differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(\text{constant}) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u'v + uv'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = \csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

If $y = [f(x)]^{g(x)}$, take the logarithm of both sides and then differentiate.

Table 1: Differentiation by substitution

| Expression | Substitution to perform |
|---|---|
| $a^2 - x^2$ | Put $x = a\sin t$ or $x = a\cos t$ |
| $a^2 + x^2$ | Put $x = a\tan t$ or $x = a\cot t$ |
| $x^2 - a^2$ | Put $x = a\sec t$ or $x = a\csc t$ |
| $\sqrt{\frac{a-x}{a+X}}$ or $\sqrt{\frac{a+x}{a-x}}$ | Put $x = a\cos t$ |
| $a\cos x \pm b\sin x$ | Put $a = r\cos\alpha$, $b = r\sin\alpha$ |

# 9 Indeterminate Forms of Limits

## 10 Mean Value Theorems

### 10.1 Rolle's Theorem

If $f(x)$ satisfies the following conditions:

- $f(x)$ is continuous in the *closed* interval $[a, b]$
- $f(x)$ is derivable in the *open* interval $(a, b)$
- $f(a) = f(b)$

... then there exists at least one real number $x = c$ in the *open* interval $(a, b)$ such that $f'(c) = 0$.

### 10.2 Langrange's Mean Value Theorem

If a function $f(x)$ defined in the closed interval $[a, b]$ satisfies the following conditions:

- $f(x)$ is continuous in the *closed* interval $[a, b]$
- $f(x)$ is derivable in the *open* interval $(a, b)$

... there exists at least one value of $x$, $c$, in the *open* interval $(a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

# 11 Applications of Derivatives

## 11.1 Tangents and Normals

$$m = \frac{\mathrm{d}y}{\mathrm{d}x}(x')$$

$$y - y' = m(x - x')$$

$$y - y' = -\frac{1}{m}(x - x')$$

## 11.2 Approximations

$$\delta y = \frac{\mathrm{d}y}{\mathrm{d}x}(x') \times \delta x$$

## 11.3 Rate Measuring

$$\text{Rate of change of } y = \frac{\mathrm{d}y}{\mathrm{d}x} \times \text{Rate of change of } x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

## 11.4 Monotonocity

$$\frac{\mathrm{d}y}{\mathrm{d}x} \begin{cases} > 0 & \tan\psi \text{ is acute} \\ < 0 & \tan\psi \text{ is obtuse} \\ = 0 & \tan\psi = \psi = 0 \\ \text{is undefined} & \psi = 90^\circ \end{cases}$$

$$f(x) \begin{cases} \text{increasing} & f'(x) > 0 \\ \text{decreasing} & f'(x) < 0 \\ \text{constant} & f'(x) = 0 \end{cases}$$

## 12 Maxima and Minima

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

| Minima | Maxima |
|---|---|
| $\frac{\mathrm{d}^2 y}{\mathrm{dx}^2} > 0$ | $\frac{\mathrm{d}^2 y}{\mathrm{dx}^2} < 0$ |
| $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from $-$ to $+$ | $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from $+$ to $-$ |

## 13 Indefinite Integrals — Standard Forms

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

$$\int (ax+b)^n \mathrm{d}x = \frac{(ax+b)^{n+1}}{(n+1)a} + C$$

$$\int \cos x \mathrm{d}x = \sin x + C$$

$$\int \sin x \mathrm{d}x = -\cos x + C$$

$$\int \sec^2 x \mathrm{d}x = \tan x + C$$

$$\int \csc^2 x \mathrm{d}x = -\cot x + C$$

$$\int \csc x \cot x \mathrm{d}x = -\csc x + C$$

$$\int \sec x \tan x \mathrm{d}x = \sec x + C$$

$$\int \tan x \mathrm{d}x = \ln|\sec x| + C$$

$$\int \cot x \mathrm{d}x = \ln|\sin x| + C$$

$$\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

## 14 Indefinite Integrals — Methods of Integration

$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \sin^{-1} x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}} \mathrm{d}x = \cos^{-1} x + C$$

$$\int \frac{1}{1+x^2} \mathrm{d}x = \tan^{-1} x + C$$

$$\int \frac{-1}{1+x^2} \mathrm{d}x = \cot^{-1} x + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} \mathrm{d}x = \sec^{-1} x + C$$

$$\int \frac{-1}{|x|\sqrt{x^2-1}} \mathrm{d}x = \csc^{-1} x + C$$

$$\int e^x [f(x) + f'(x)] \mathrm{d}x = e^x \cdot f(x) + C$$

$$\int (f \cdot g) \mathrm{d}x = f \int g \mathrm{d}x - \int \left[ f' \int g \mathrm{d}x \right] \mathrm{d}x$$

## 15 Indefinite Integrals — Special Integrals

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \sqrt{x^2\pm a^2}\mathrm{d}x = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right| + C$$

$$\int \sqrt{a^2-x^2}\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

### 15.1 $\int \sqrt{ax^2+bx+c}\mathrm{d}x$, $\int (px+q)\sqrt{ax^2+bx+c}\mathrm{d}x$

### 15.2 $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\mathrm{d}x$

Let $A$ and $B$ be constants. If $N$ is the numerator and $D$ is the denominator, set:

$$N = A(D) + B\frac{\mathrm{d}}{\mathrm{d}x}(D)$$

### 15.3 $\int \frac{x^2\pm 1}{x^4+kx^2+1}\mathrm{d}x$ ($k$ may be $0$)

Divide numerator and denominator by $x^2$. Let $t = x + \frac{1}{x}$ or $t = x - \frac{1}{x}$ depending on which $t$ gives the numerator of the resulting integrand on differentiation.

**15.4** $\int \frac{\mathrm{d}x}{a+b\cos x}$, $\int \frac{\mathrm{d}x}{a+b\sin x}$, $\int \frac{\mathrm{d}x}{a\cos x+b\sin x}$, $\int \frac{\mathrm{d}x}{a\cos x+b\sin x+c}$

- Substitute $\sin x = \frac{2\tan\frac{x}{2}}{a+\tan^2\frac{x}{2}}$, $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.
- Replace $1+\tan^2\frac{x}{2}$ in the numerator with $\sec^2\frac{x}{2}$.
- Substitute $t = \tan\frac{x}{2}$, $\mathrm{d}t = \frac{1}{2}\sec^2\frac{x}{2}$

**15.5** $\int \frac{\mathrm{d}x}{a\cos^2 x+b\sin^2 x+c}$

**15.6** $\int \frac{\mathrm{d}x}{1+x^4}$

**15.7** $\int \sqrt{\tan x}\mathrm{d}x$, $\int \sqrt{\cot x}\mathrm{d}x$

## 16 Definite Integrals

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t$$

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$$

$$\int_a^a f(x)\mathrm{d}x = 0$$

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$$

$$\int_0^a f(x)\mathrm{d}x = \int_0^a f(a-x)\mathrm{d}x$$

$$\int_0^{2a} f(x)\mathrm{d}x = \int_0^a f(x)\mathrm{d}x + \int_0^a f(2a-x)\mathrm{d}x$$

$$\int_0^{2a} f(x)\mathrm{d}x = \begin{cases} 2\int_0^a f(x)\mathrm{d}x & f(2a-x) = f(x) \\ 0 & f(2a-x) = -f(x) \end{cases}$$

$$\int_{-a}^a f(x)\mathrm{d}x = \begin{cases} 2\int_0^a f(x)\mathrm{d}x & f(-x) = f(x) \\ 0 & f(-x) = -f(x) \end{cases}$$

## 17 Definite Integral as a limit of a sum

$$\int_a^b f(x)\mathrm{d}x = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right]$$

$$nh = b - a$$

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

$$\sin a+\sin(a+h)+\sin(a+2h)+\cdots+\sin\left(a+(n-1)h\right)=\frac{\sin\left\{a+\frac{n-1}{2}h\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$\cos a+\cos(a+h)+\cos(a+2h)+\cdots+\cos\left(a+(n-1)h\right)=\frac{\cos\left\{a+\frac{n-1}{2}h\right\}\cos\frac{nh}{2}}{\cos\frac{h}{2}}$$

# 18 Differential Equations

## 18.1 Variable Separable

### 18.1.1 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

### 18.1.2 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$

### 18.1.3 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \cdot \phi(y)$

### 18.1.4 $\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c)$

## 18.2 Reducible to Variable Separable

## 18.3 Homogeneous equations of first order

### 18.3.1 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x,y)}{g(x,y)} = m\left(\frac{y}{x}\right)$

$$y = vx$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

## 18.4 Linear equations

### 18.4.1 $\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$

$$\mathrm{IF} = e^{\int P \mathrm{d}x}$$

$$y \cdot \mathrm{IF} = \int (Q \cdot \mathrm{IF})\,\mathrm{d}x + C$$

## Part IV

# Probability

## 19 Probability

$$P(\overline{A}) = 1 - P(A)$$

$$P(A/B) = \frac{n(A \cap B)}{n(B)}$$

$$P(A \cap B) = P(B) \times P(A/B) = P(A) \times P(B/A)$$

### 19.1 $A$ and $B$ are mutually exclusive

$$P(A \cap B) = 0$$

### 19.2 $A$ and $B$ are independent events

$$P(A/B) = P(A)$$

$$P(B/A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

### 19.3 Geometric Progression

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}$$

## 20 Bayes' Theorem

$$P(E_1/A) = \frac{P(E_!) \cdot P(A/E_1)}{P(E_!) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$P(E_1/A) = \frac{P(E_!) \cdot P(A/E_1)}{P(E_!) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

# 21 Theoretical Probability Distribution

$$\int_{-\infty}^{\infty} f(x) \cdot \mathrm{d}x = 1$$

$$\overline{x} = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} x_i f(x)$$

$$\sigma^2 = \sum_{i=1}^{n} p_i \cdot x_i^2 - \overline{x}^2 = \sum_{i=1}^{n} p_i (x_i - \overline{x})^2$$

$$\sigma = +\sqrt{\sigma^2}$$

## 21.1 Binomial Distribution

$$P(r) = \binom{n}{r} q^{n-r} p^r$$

$$\mu = np$$

$$\sigma^2 = npq$$

## Part V

# Vectors

## 22 Vectors

## 23 Vectors (Continued)

## Part VI

# Three-Dimensional Geometry

## 24 Three-Dimensional Geometry

## 25 The Plane

## Part VII

# Application of Integrals

## 26 Areas of a Curve

# Part VIII

# Application of Calculus

## 27 Application of Calculus in Commerce and Economics

$$C(x) = F + V(X)$$

$$\text{AC} = \frac{C(x)}{x} = \text{AFC} + \text{AVC}$$

$$R = px$$

$$P(x) = R(x) - C(x)$$

At the break-even point (say $x_b$),

$$R(x_b) = C(x_b)$$

$$\text{MC} = \frac{\mathrm{d}}{\mathrm{d}x}\left[C(x)\right] = \frac{\mathrm{d}C}{\mathrm{d}x}$$

$$\text{AFC} = \frac{\text{TFC}}{Q}$$

$$\text{AVC} = \frac{\text{TVC}}{Q}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\text{AC}) = \frac{1}{x}(\text{MC} - \text{AC})$$

$$C = \int \mathrm{MC} \cdot \mathrm{d}x$$

$$C_d = \int_0^d \mathrm{MC} \cdot \mathrm{d}x$$

$$R = \int \mathrm{MR} \cdot \mathrm{d}x$$

$$R_d = \int_0^d \mathrm{MR} \cdot \mathrm{d}x$$

# Part IX

# Linear Regression

## 28 Linear Regression

*dependent* on *independent*

$$d_x = x_i - \overline{x}$$

$$d_y = y_i - \overline{y}$$

| | $y$ on $x$ | $x$ on $y$ |
|---|---|---|
| Regression line | $y - \overline{y} = b_{yx}(x - \overline{x})$ | $x - \overline{x} = b_{xy}(y - \overline{y})$ |
| Normal Equations | $\sum y = nc + m\sum x$ | $\sum x = nc + m\sum y$ |
| | $\sum xy = c\sum x + m\sum x^2$ | $\sum xy = c\sum y + m\sum y^2$ |
| Regression Coefficient | $b_{yx} = r\frac{\sigma_y}{\sigma_x}$ | $b_{xy} = r\frac{\sigma_x}{\sigma_y}$ |
| When deviations are taken from the mean | $b_{yx} = \frac{\Sigma d_x d_y}{\Sigma d_x^2}$ | $b_{xy} = \frac{\Sigma d_x d_y}{\Sigma d_y^2}$ |
| When deviations are taken from the assumed mean | $b_{yx} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}}{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{n}}$ | $b_{xy} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}}{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{n}}$ |
| When original values are used | $b_{yx} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$ | $b_{xy} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}$ |

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

$$r^2 \leq 1$$

$$r = \frac{\sum (d_x d_y)}{\sqrt{\sum d_x^2 \times \sum d_x^2}}$$

Point of intersection of the two regression lines is $(\overline{x}, \overline{y})$.

# Part X

# Linear Programming

## 29 Linear Programming

**Part XI**

# Appendices

## A Useful Trigonometric Formulae

## B Other Useful Formulae

$$a^3 \pm b^3 = (a \pm b)(a^2 + ab + b^2)$$